What is a barycentric bear?

https://varamu.eu

14.06.2024

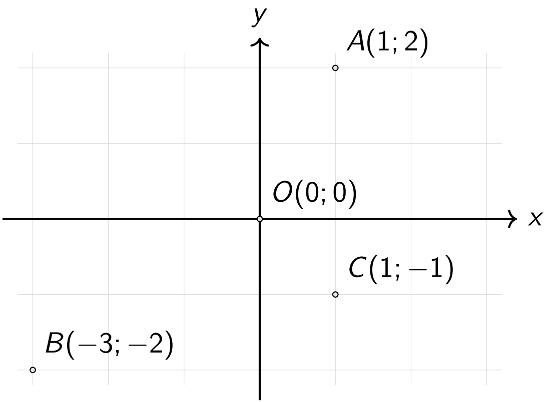
What is a polar bear?

What is a polar bear?

- It is a usual rectangular bear transformed to polar coordinates.

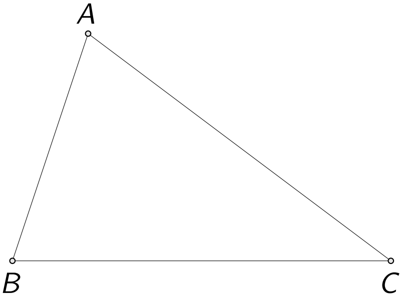
Cartesian coordinates

- are based on two perpendicular straight lines (called *axes*):



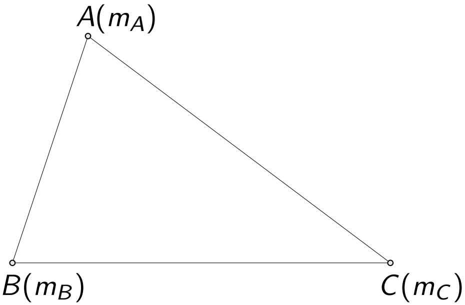
Barycentric coordinates

- are based on a triangle:



Barycentric coordinates

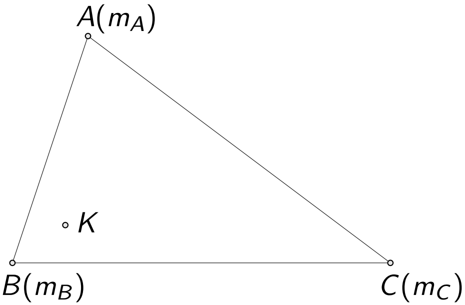
- are based on a triangle:



- Apply masses m_A, m_B, m_C in the vertices A, B, C, respectively.
  - The masses may be negative.
  - However, we require $m_A + m_B + m_C \neq 0$.

Barycentric coordinates

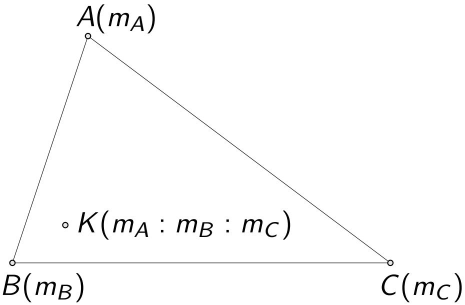
- are based on a triangle:



- Apply masses m_A, m_B, m_C in the vertices A, B, C, respectively.
  - The masses may be negative.
  - However, we require $m_A + m_B + m_C \neq 0$.
- Let the mass center of the point system $\{A, B, C\}$ be K.

Barycentric coordinates

- are based on a triangle:



- Apply masses m_A, m_B, m_C in the vertices A, B, C, respectively.
  - The masses may be negative.
  - However, we require $m_A + m_B + m_C \neq 0$.
- Let the mass center of the point system $\{A, B, C\}$ be K.
- The triple $(m_A : m_B : m_C)$ is called *barycentric coordinates of K w.r.t. the triangle ABC*.

Barycentric coordinates are homogeneous

- Note that barycentric coordinates of a point are not unique.
- For any $t \neq 0$, the coordinates $(m_A : m_B : m_C)$ and $(tm_A : tm_B : tm_C)$ define the same point.
  - We say that a coordinate system with this property is *homogeneous*.

Barycentric coordinates are homogeneous

- Note that barycentric coordinates of a point are not unique.
- For any $t \neq 0$, the coordinates $(m_A : m_B : m_C)$ and $(tm_A : tm_B : tm_C)$ define the same point.
  - We say that a coordinate system with this property is *homogeneous*.
- However, if we additionally require that $m_A + m_B + m_C = 1$, each point on the plane has unique barycentric coordinates.
- Such coordinates are called *normalized*.

Let's find some barycentric coordinates

- A
- B
- C

Let's find some barycentric coordinates

- $A(1:0:0)$
- $B(0:1:0)$
- $C(0:0:1)$

Let's find some barycentric coordinates

- $A(1 : 0 : 0)$
- $B(0 : 1 : 0)$
- $C(0 : 0 : 1)$
- Centroid M

Let's find some barycentric coordinates

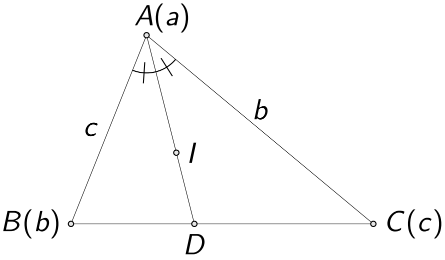
- $A(1:0:0)$
- $B(0:1:0)$
- $C(0:0:1)$
- Centroid $M(1:1:1) = M(\frac{1}{3}:\frac{1}{3}:\frac{1}{3})$

Let's find some barycentric coordinates

- $A(1 : 0 : 0)$
- $B(0 : 1 : 0)$
- $C(0 : 0 : 1)$
- Centroid $M(1 : 1 : 1) = M(\frac{1}{3} : \frac{1}{3} : \frac{1}{3})$
- Incenter I

Let's find some barycentric coordinates

- $A(1:0:0)$
- $B(0:1:0)$
- $C(0:0:1)$
- Centroid $M(1:1:1) = M(\frac{1}{3}:\frac{1}{3}:\frac{1}{3})$
- Incenter $I(a:b:c)$, where a, b, c are side lengths of the triangle ABC:



$$\frac{|BD|}{|CD|} = \frac{c}{b}$$

Some more barycentric coordinates

- Excenter $I_a(-a : b : c)$
- Orthocenter $H(\tan\alpha : \tan\beta : \tan\gamma)$
- Circumcenter $O(\sin 2\alpha : \sin 2\beta : \sin 2\gamma)$

Some more barycentric coordinates

- Excenter $I_a(-a : b : c)$
- Orthocenter $H(\tan\alpha : \tan\beta : \tan\gamma)$
- Circumcenter $O(\sin 2\alpha : \sin 2\beta : \sin 2\gamma)$
- To compare, in Cartesian coordinates, the orthocenter and a circumcenter of a triangle with the vertices $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$ are

$$H\left(\frac{x_A\tan\alpha + x_B\tan\beta + x_C\tan\gamma}{\tan\alpha+\tan\beta+\tan\gamma}, \frac{y_A\tan\alpha + y_B\tan\beta + y_C\tan\gamma}{\tan\alpha+\tan\beta+\tan\gamma}\right) \text{ and}$$

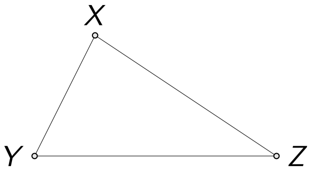
$$O\left(\frac{x_A\sin 2\alpha + x_B\sin 2\beta + x_C\sin 2\gamma}{\sin 2\alpha+\sin 2\beta+\sin 2\gamma}, \frac{y_A\sin 2\alpha + y_B\sin 2\beta + y_C\sin 2\gamma}{\sin 2\alpha+\sin 2\beta+\sin 2\gamma}\right)$$

- Essentially, you first find H and O in barycentric coordinates and then translate them to the Cartesian representation!
- So why not using the barycentric coordinates all along?

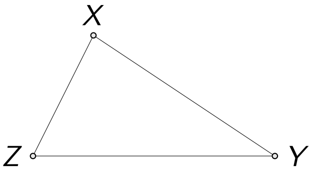
Barycetric coordinates and areas

Definition

Let $[XYZ]$ denote *signed* area of a triangle XYZ.



$[XYZ] = S_{XYZ}$

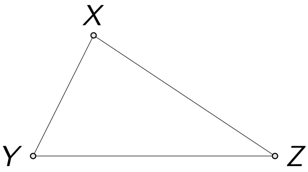


$[XYZ] = -S_{XYZ}$

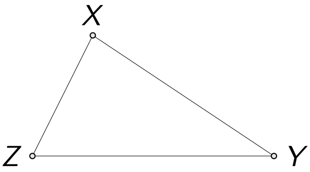
Barycetric coordinates and areas

Definition

Let $[XYZ]$ denote *signed* area of a triangle XYZ.



$[XYZ] = S_{XYZ}$



$[XYZ] = -S_{XYZ}$

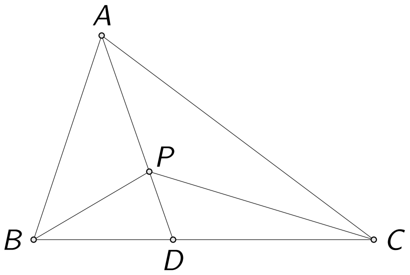
Theorem

Any point P on the plane has normalized barycentric (or *areal*) coordinates

$$\frac{1}{[ABC]}([PBC] : [PCA] : [PAB]) .$$

Proof idea

- Consider the case when P lies inside the triangle ABC



- Then we have

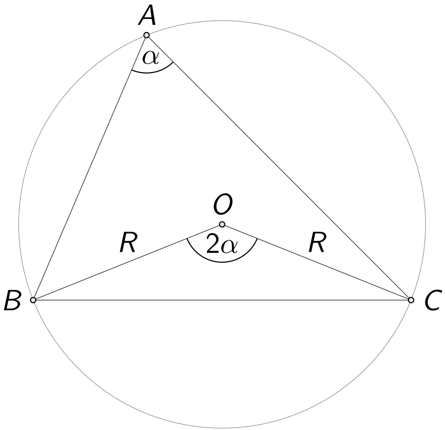
$$\frac{S_{ABD}}{S_{ADC}} = \frac{|BD|}{|DC|} \quad \text{and} \quad \frac{S_{PBD}}{S_{PDC}} = \frac{|BD|}{|DC|},$$

thus also

$$\frac{S_{PAB}}{S_{PCA}} = \frac{S_{ABD} - S_{PBD}}{S_{ADC} - S_{PDC}} = \frac{|BD|}{|DC|}.$$

- The other cases are similar :)

$O(\frac{1}{2}R^2 \sin 2\alpha : \frac{1}{2}R^2 \sin 2\beta : \frac{1}{2}R^2 \sin 2\gamma) = (\sin 2\alpha : \sin 2\beta : \sin 2\gamma)$



$$[OBC] = \frac{1}{2}R^2 \sin 2\alpha$$

Area and collinearity

Theorem

Consider three points on the plane given by their normalized barycentric coordinates $P(x_1 : y_1 : z_1)$, $Q(x_2 : y_2 : z_2)$, $R(x_3 : y_3 : z_3)$. Then

$$[PQR] = [ABC] \cdot \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Area and collinearity

Theorem

Consider three points on the plane given by their normalized barycentric coordinates $P(x_1 : y_1 : z_1)$, $Q(x_2 : y_2 : z_2)$, $R(x_3 : y_3 : z_3)$. Then

$$[PQR] = [ABC] \cdot \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} .$$

- The points P, Q, R are collinear iff $[PQR] = 0$.
- Thus, in barycentric coordinates, the collinearity condition is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0 .$$

This holds also for non-normalized coordinates!

Equation of a line

Theorem

Equation of the line passing through $P(x_1 : y_1 : z_1)$ and $Q(x_2 : y_2 : z_2)$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x & y & z \end{vmatrix} = 0\,.$$

In general, equation of a line is

$$ux + vy + wz = 0$$

for real u, v, w.

Example: Euler line

- The line through the centroid $M(1:1:1)$ and circumcenter $O(\sin 2\alpha : \sin 2\beta : \sin 2\gamma)$ is

$$\begin{vmatrix} 1 & 1 & 1 \\ \sin 2\alpha & \sin 2\beta & \sin 2\gamma \\ x & y & z \end{vmatrix} = 0 \quad \text{or}$$

$$(\sin 2\gamma - \sin 2\beta)x + (\sin 2\alpha - \sin 2\gamma)y + (\sin 2\beta - \sin 2\alpha)z = 0\,.$$

Example: Euler line

- The line through the centroid $M(1:1:1)$ and circumcenter $O(\sin 2\alpha : \sin 2\beta : \sin 2\gamma)$ is

$$\begin{vmatrix} 1 & 1 & 1 \\ \sin 2\alpha & \sin 2\beta & \sin 2\gamma \\ x & y & z \end{vmatrix} = 0 \quad \text{or}$$

$$(\sin 2\gamma - \sin 2\beta)x + (\sin 2\alpha - \sin 2\gamma)y + (\sin 2\beta - \sin 2\alpha)z = 0\,.$$

- Recall: $\sin\varphi - \sin\psi = 2\cos(\frac{\varphi+\psi}{2})\sin(\frac{\varphi-\psi}{2})$, so

$$\sin 2\gamma - \sin 2\beta = 2\cos(\gamma+\beta)\sin(\gamma-\beta) = -2\cos\alpha\sin(\gamma-\beta)\,,$$

since $\alpha + \beta + \gamma = \pi$.

Example: Euler line

- The line through the centroid $M(1:1:1)$ and circumcenter $O(\sin 2\alpha : \sin 2\beta : \sin 2\gamma)$ is

$$\begin{vmatrix} 1 & 1 & 1 \\ \sin 2\alpha & \sin 2\beta & \sin 2\gamma \\ x & y & z \end{vmatrix} = 0 \quad \text{or}$$

$$(\sin 2\gamma - \sin 2\beta)x + (\sin 2\alpha - \sin 2\gamma)y + (\sin 2\beta - \sin 2\alpha)z = 0\,.$$

- Recall: $\sin\varphi - \sin\psi = 2\cos(\frac{\varphi+\psi}{2})\sin(\frac{\varphi-\psi}{2})$, so

$$\sin 2\gamma - \sin 2\beta = 2\cos(\gamma+\beta)\sin(\gamma-\beta) = -2\cos\alpha\sin(\gamma-\beta)\,,$$

since $\alpha+\beta+\gamma=\pi$.

- Thus, we can rewrite the equation of the line as

$$\cos\alpha\sin(\gamma-\beta)x + \cos\beta\sin(\alpha-\gamma)y + \cos\gamma\sin(\beta-\alpha)z = 0\,.$$

Example (cont'd): Orthocenter lies on the Euler line

- Recall: $H(\tan\alpha : \tan\beta : \tan\gamma)$. Thus, we must evaluate

$$E = \cos\alpha\sin(\gamma-\beta)\tan\alpha + \cos\beta\sin(\alpha-\gamma)\tan\beta + \cos\gamma\sin(\beta-\alpha)\tan\gamma\,.$$

Example (cont'd): Orthocenter lies on the Euler line

- Recall: $H(\tan\alpha : \tan\beta : \tan\gamma)$. Thus, we must evaluate

$$E = \cos\alpha\sin(\gamma-\beta)\tan\alpha + \cos\beta\sin(\alpha-\gamma)\tan\beta + \cos\gamma\sin(\beta-\alpha)\tan\gamma\,.$$

- Simplify the first term:

$$\cos\alpha\sin(\gamma-\beta)\tan\alpha = \sin\alpha\sin(\gamma-\beta) = \sin(\gamma+\beta)\sin(\gamma-\beta)\,.$$

Example (cont'd): Orthocenter lies on the Euler line

- Recall: $H(\tan\alpha : \tan\beta : \tan\gamma)$. Thus, we must evaluate

$$E = \cos\alpha\sin(\gamma-\beta)\tan\alpha + \cos\beta\sin(\alpha-\gamma)\tan\beta + \cos\gamma\sin(\beta-\alpha)\tan\gamma\,.$$

- Simplify the first term:

$$\cos\alpha\sin(\gamma-\beta)\tan\alpha = \sin\alpha\sin(\gamma-\beta) = \sin(\gamma+\beta)\sin(\gamma-\beta)\,.$$

- Recall: $\sin\varphi\sin\psi = \frac{1}{2}(\cos(\varphi-\psi) - \cos(\varphi+\psi))$, thus

$$\sin(\gamma+\beta)\sin(\gamma-\beta) = \frac{1}{2}(\cos 2\beta - \cos 2\gamma) \quad \text{and}$$

$$E = \frac{1}{2}(\cos 2\beta - \cos 2\gamma) + \frac{1}{2}(\cos 2\gamma - \cos 2\alpha) + \frac{1}{2}(\cos 2\alpha - \cos 2\beta) = 0\,.$$

Euler line in the nature